REPORT 1059

A BIHARMONIC RELAXATION METHOD FOR CALCULATING THERMAL STRESS IN COOLED IRREGULAR CYLINDERS[1]

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SUMMARY

A numerical method was developed for calculating thermal stresses in irregular cylinders cooled by one or more internal passages. The use of relaxation methods and elementary methods of finite differences was found to give approximations to the correct values when compared with previously known solutions for concentric circular cylinders possessing symmetrical and asymmetrical temperature distributions.

INTRODUCTION

USE OF COOLED IRREGULAR CYLINDERS

The evolution of aircraft propulsion systems has led to the frequent employment of cooled structures. The concentric hollow cylinder is a familiar example, although in some cases irregular cylinders such as cooled turbine blades with several internal passages (reference 1) are under consideration. A method of calculating stresses in thin-walled turbine blades of the air-cooled type is presented in reference 2, but the general problem of calculating thermal stresses in long, hollow, thick-walled irregular cylinders has not been solved.

PREVIOUS WORK ON THERMAL STRESSES IN HOLLOW CYLINDERS

Several methods of calculating thermal stresses for various special distributions of temperature in long hollow cylinders of particular shape have been developed. Some theoretical aspects of the general problem have also been discussed. In reference 3 the problem is regarded as an ordinary stress problem with given body and surface forces replacing the effects due to temperature distribution, whereas in reference 4 the equilibrium and boundary conditions of the theory of elasticity are used without modifications to exhibit the temperature effects as body and surface forces. The method of reference 4 was applied to several special problems that had already been solved as well as to problems of composite bodies and eccentric circular cylinders. No application of analytical or numerical methods of calculating thermal stresses in cooled irregular multiply connected cylinders has been published.

SCOPE OF PRESENT INVESTIGATION

An investigation was conducted at the NACA Lewis laboratory during 1949–50 to calculate thermal stresses in cooled irregular multiply connected cylinders. The problem of thermal stresses in irregular cylinders is formulated in a manner that permits solution by the use of finite-difference methods. The contour integrals of reference 5 expressing the single-valued character of the displacements for arbitrary circuits around the internal boundaries are written in forms suitable for numerical methods of differentiation and integration. The boundary conditions based on the assumption of force-free boundaries and single-valued displacements are formulated in terms of derivatives of the stress function as suggested for uniform-temperature problems in reference 5, and stress functions are set up in a manner that is an extension to the thermal-stress problem of the work of reference 6 on doubly connected domains at uniform temperatures. The relaxation techniques of reference 7 are used in solving the finite-difference problem of determining the stress functions. Details of the method are illustrated by examples. The method is applied to a symmetrically heated, hollow circular cylinder and also to a hollow circular cylinder with asymmetrical heating to show that the relaxation technique gives approximations to exact answers obtained by direct mathematical methods. Comparison of stresses calculated by the relaxation techniques with those determined by exact methods is also made to compare the relative importance of several sources of error. The calculations for the concentric cylinder are described in detail sufficient to permit the method to be applied to more irregular cylinders.

SYMBOLS

The following symbols are used in this report:

a_{11},a_{12},a_{13}	constants of integration
E	modulus of elasticity in tension and compression
l,m	direction cosines of normal drawn outward from region bounded by plane curve
n	distance in xy-plane normal to plane curve
Q	residual in relaxation calculation
r,θ	polar coordinates
s	arc length of plane curve in xy-plane
T	temperature above initial stress-free state
u,v,w	components of displacement
x,y,z	rectangular coordinates
α	coefficient of linear thermal expansion
$\gamma_{xy},\gamma_{xz},\gamma_{yz}$	shearing strain components in rectangular coordinates
$\epsilon_x,\epsilon_y,\epsilon_z$	unit elongations (strains) in x-, y-, and z-directions, respectively
ν	Poisson's ratio

[1] Supersedes NACA TN 2434, "A Biharmonic Relaxation Method for Calculating Thermal Stress in Cooled Irregular Cylinders" by Arthur G. Holms, 1951.

$\sigma_x, \sigma_y, \sigma_z$	normal components of stress parallel to x-, y-, and z-axes, respectively
τ_{xy}	shearing-stress component in rectangular coordinates
ϕ	Airy's stress function
ω	component of rotation about z-axis
∇^2	harmonic operator $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$
∇^4	biharmonic operator $\frac{\partial^4}{\partial x^4}+2\frac{\partial^4}{\partial x^2 y^2}+\frac{\partial^4}{\partial y^4}$

THEORY

ASSUMPTIONS

The following conditions are assumed to exist:

1. Steady-state heat flow exists with temperatures uniform along any line initially parallel to the axis of the cylinder.
2. Heat sources and sinks are assumed to be distributed on the external and internal boundaries.
3. The temperature distribution is assumed to be determined by the boundary temperatures and Laplace's equation.
4. The material behaves in an elastic manner.
5. The variation of the elastic constants (modulus of elasticity, Poisson's ratio, and coefficient of thermal expansion) with temperature may be neglected in determining the thermal stresses.
6. Plane sections initially normal to the axis of the cylinder remain plane.
7. The strains and rotation are constant along any line initially parallel to the axis of the cylinder.

The extent to which a particular structure would fulfill these conditions would depend on the particular circumstances. The last two assumptions are appropriate when the cylinder is long in comparison with its cross-sectional dimensions or when end conditions impose suitable restraint. The sixth assumption allows bending of the cylinder about axes perpendicular to the axial direction of the cylinder in a manner that might vary in the axial direction. The last assumption permits planes initially perpendicular to the axis of the cylinder to take on a warped shape as a result of the deformation but restricts the bending to a circular arc; that is, the radius of curvature of lines initially parallel to the axis of the cylinder does not vary along the length of these lines. The assumption of no variation of rotation in the axial direction is equivalent to assuming that the cylinder does not twist.

BASIC EQUATIONS

The case of plane strain where body forces are given by the gradient of a potential is treated in reference 5. As shown by the detailed derivation in appendix A, the governing equation of reference 5 applies to the thermal-stress problem defined by the preceding assumptions. This equation is

$$\nabla^4\phi=0 \tag{1}$$

where ϕ is Airy's stress function defined by

$$\left.\begin{aligned} \sigma_x&=\frac{\partial^2\phi}{\partial y^2} \\ \sigma_y&=\frac{\partial^2\phi}{\partial x^2} \\ \tau_{xy}&=-\frac{\partial^2\phi}{\partial x\partial y} \end{aligned}\right\} \tag{2}$$

As shown in appendix B, the axial stresses can be calculated from the equation

$$\sigma_z=E(ax+by+c)+\nu(\sigma_x+\sigma_y)-\alpha ET \tag{3}$$

BOUNDARY CONDITIONS

Two basically different types of auxiliary condition are established for the set of physical conditions assumed to exist. These conditions are:

(1) Conditions stemming from the nature of the forces applied to the surfaces

(2) Conditions stipulating that the displacements be single valued

In calculating the stresses due to the temperature distribution, other stresses such as those due to centrifugal force and fluid pressure are to be calculated separately and the total stresses are to be obtained by superposition. All internal and external boundaries are therefore postulated to be free from applied forces, and the boundary conditions (reference 8, p. 21) become

$$\left.\begin{aligned} \sigma_x l+\tau_{xy}m&=0 \\ \tau_{xy}l+\sigma_y m&=0 \end{aligned}\right\} \tag{4}$$

where

$$l=\frac{dy}{ds}$$

$$m=-\frac{dx}{ds}$$

Substitution from equations (2) into equations (4) yields

$$\frac{\partial^2\phi}{\partial y^2}\frac{dy}{ds}+\frac{\partial^2\phi}{\partial x\partial y}\frac{dx}{ds}=0$$

$$\frac{\partial^2\phi}{\partial x\partial y}\frac{dy}{ds}+\frac{\partial^2\phi}{\partial x^2}\frac{dx}{ds}=0$$

from which

$$\left.\begin{aligned} \frac{d}{ds}\frac{\partial\phi}{\partial y}&=0 \\ \frac{d}{ds}\frac{\partial\phi}{\partial x}&=0 \end{aligned}\right\} \tag{5}$$

along the boundaries.

CONDITIONS FOR SINGLE-VALUED DISPLACEMENTS

The defining equations for strains and rotation are

$$\left.\begin{aligned}\epsilon_x&=\frac{\partial u}{\partial x}\\ \epsilon_y&=\frac{\partial v}{\partial y}\\ \gamma_{xy}&=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\\ \omega&=\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\end{aligned}\right\} \tag{6}$$

From equations (6),

$$\left.\begin{aligned}\frac{\partial u}{\partial x}&=\epsilon_x\\ \frac{\partial v}{\partial x}&=\frac{1}{2}\gamma_{xy}+\omega\\ \frac{\partial u}{\partial y}&=\frac{1}{2}\gamma_{xy}-\omega\\ \frac{\partial v}{\partial y}&=\epsilon_y\\ \frac{\partial \omega}{\partial x}&=\frac{1}{2}\frac{\partial \gamma_{xy}}{\partial x}-\frac{\partial \epsilon_x}{\partial y}\\ \frac{\partial \omega}{\partial y}&=-\frac{1}{2}\frac{\partial \gamma_{xy}}{\partial y}+\frac{\partial \epsilon_y}{\partial x}\end{aligned}\right\} \tag{7}$$

The change in rotation for an arbitrary path P_i of integration starting at some point (x_0, y_0) and returning to the same point after enclosing (and only enclosing) the internal boundary C_i is

$$\Big[\omega\Big]_0^p=\oint_{P_i} d\omega=\oint_{P_i}\frac{\partial \omega}{\partial x}\,dx+\frac{\partial \omega}{\partial y}\,dy \tag{8}$$

From equations (7) and (8), the condition that ω be single valued is

$$\Big[\omega\Big]_0^p=\oint_{P_i}\left[\left(\frac{1}{2}\frac{\partial \gamma_{xy}}{\partial x}-\frac{\partial \epsilon_x}{\partial y}\right)dx+\left(-\frac{1}{2}\frac{\partial \gamma_{xy}}{\partial y}+\frac{\partial \epsilon_y}{\partial x}\right)dy\right]=0 \tag{9}$$

By use of equations (A12) and (A13) of appendix A, equation (9) may be written

$$\oint_{P_i}\frac{\partial \nabla^2\phi}{\partial n}\,ds=-\frac{\alpha E}{1-\nu}\oint_{P_i}\frac{\partial T}{\partial n}\,ds \tag{10}$$

The change in the x-component of displacement for an arbitrary path of integration starting at some point (x_0, y_0) and returning to the same point after enclosing the internal boundary C_i is

$$\Big[u\Big]_0^p=\oint_{P_i} du=\oint_{P_i}\frac{\partial u}{\partial x}\,dx+\frac{\partial u}{\partial y}\,dy \tag{11}$$

From equations (7) and (11), the condition for single-valued u is

$$\oint_{P_i}\epsilon_x dx+\left(\frac{1}{2}\gamma_{xy}-\omega\right)dy=0 \tag{12}$$

For the term involving rotation in equation (12), integration by parts gives

$$-\oint_{P_i}\omega\,dy=-\Big[\omega y\Big]_0^p+\oint_{P_i}y\,d\omega$$

where

$$\Big[\omega y\Big]_0^p=0$$

because of the single-valued character of ω achieved by imposing equation (10). Furthermore,

$$\begin{aligned}\oint_{P_i}y\,d\omega&=\oint_{P_i}y\left(\frac{\partial \omega}{\partial x}dx+\frac{\partial \omega}{\partial y}dy\right)\\ &=\oint_{P_i}y\left(\frac{1}{2}\frac{\partial \gamma_{xy}}{\partial x}-\frac{\partial \epsilon_x}{\partial y}\right)dx+\oint_{P_i}y\left(-\frac{1}{2}\frac{\partial \gamma_{xy}}{\partial y}+\frac{\partial \epsilon_y}{\partial x}\right)dy\end{aligned}$$

Equation (12) can now be written

$$\oint_{P_i}\left(\epsilon_x dx+\frac{1}{2}\gamma_{xy}dy+\frac{1}{2}y\frac{\partial \gamma_{xy}}{\partial x}dx-\right.$$
$$\left.y\frac{\partial \epsilon_x}{\partial y}dx-\frac{1}{2}y\frac{\partial \gamma_{xy}}{\partial y}dy+y\frac{\partial \epsilon_y}{\partial x}dy\right)=0 \tag{13}$$

In equation (13), integration of the first two terms by parts gives

$$\oint_{P_i}\left(\epsilon_x dx+\frac{1}{2}\gamma_{xy}dy\right)=\Big[x\,\epsilon_x\Big]_0^p+\frac{1}{2}\Big[y\gamma_{xy}\Big]_0^p-$$
$$\oint_{P_i}x\left(\frac{\partial \epsilon_x}{\partial x}dx+\frac{\partial \epsilon_x}{\partial y}dy\right)-\frac{1}{2}\oint_{P_i}y\left(\frac{\partial \gamma_{xy}}{\partial x}dx+\frac{\partial \gamma_{xy}}{\partial y}dy\right)$$

where

$$\Big[x\,\epsilon_x\Big]_0^p=\Big[y\,\gamma_{xy}\Big]_0^p=0$$

because the strains are single valued. Equation (13) becomes

$$\oint_{P_i}\left(-y\frac{\partial \epsilon_x}{\partial y}-x\frac{\partial \epsilon_x}{\partial x}\right)dx+\oint_{P_i}\left(-y\frac{\partial \gamma_{xy}}{\partial y}+y\frac{\partial \epsilon_y}{\partial x}-x\frac{\partial \epsilon_x}{\partial y}\right)dy=0 \tag{14}$$

Substituting from equations (A12) and (A13) of appendix A and stipulating that the stresses are single valued give equation (14) as

$$\oint_{P_i}\left(y\frac{\partial \nabla^2\phi}{\partial n}-x\frac{d\nabla^2\phi}{ds}\right)ds=-\frac{\alpha E}{1-\nu}\oint_{P_i}y\left(\frac{\partial T}{\partial n}-x\frac{dT}{ds}\right)ds+$$
$$\frac{1}{1-\nu}\oint_{P_i}\frac{d}{ds}\frac{\partial \phi}{\partial x}ds \tag{15}$$

If the integrations of equation (15) are performed along the internal boundary C_i, then from equations (5),

$$\frac{1}{1-\nu}\oint_{C_i}\frac{d}{ds}\frac{\partial\phi}{\partial x}ds=0$$

Because the integrand of the preceding integral is an exact differential, the integral has the same value (zero) for all reducible paths; hence for the arbitrary circuit P_i equation (15) becomes

$$\oint_{P_i}\left(y\frac{\partial\nabla^2\phi}{\partial n}-x\frac{d\nabla^2\phi}{ds}\right)ds=-\frac{\alpha E}{1-\nu}\oint_{P_i}\left(y\frac{\partial T}{\partial n}-x\frac{dT}{ds}\right)ds \tag{16}$$

Similar reasoning from the single-valued character of the y-component of displacement leads to the equation

$$\oint_{P_i}\left(y\frac{d\nabla^2\phi}{ds}+x\frac{\partial\nabla^2\phi}{\partial n}\right)ds=-\frac{\alpha E}{1-\nu}\oint_{P_i}\left(y\frac{dT}{ds}+x\frac{\partial T}{\partial n}\right)ds \tag{17}$$

Equations (10), (16), and (17) are similar in form to some equations derived by Mindlin in reference 9. The equations of that reference apply to integration paths taken along the boundaries.

BOUNDARY CONSTANTS

The initial objective is to determine the stress function ϕ so as to solve the boundary-value problem associated with equation (1). Numerical differentiation could then be performed to determine the stress components according to equations (2).

The assumption of stress-free surfaces resulted in equations (5). With a_{i1} and a_{i2} as constants of integration, these boundary conditions may be integrated to give along the boundaries

$$\left.\begin{aligned}\frac{\partial\phi}{\partial x}&=a_{i1}\\ \frac{\partial\phi}{\partial y}&=a_{i2}\end{aligned}\right\} \tag{18}$$

In general,

$$d\phi=\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy$$

and where a_{i3} is a constant of integration, integration of the preceding equation along a boundary and use of equations (18) give

$$\phi_i=a_{i1}x+a_{i2}y+a_{i3} \tag{19}$$

along a boundary.

Differential equation (1) and boundary conditions, expressible by equations (18) and (19), do not completely determine the stress distribution. No temperature terms are present, and the values of a_{i1}, a_{i2}, and a_{i3} are still to be specified. The initial step in obtaining the compatibility condition (equation (1)) was to raise the order of the first differential equation of equations (A2) by differentiating once with respect to x and once with respect to y. These differentiations, although useful in simplifying the final form of the equation, require that additional factors be considered. In the case of multiply connected regions, the resulting equation (equation (1)) does not preclude the occurrence of stress distributions due to mechanical dislocations (reference 9). The occurrence of mechanical dislocations is eliminated by imposing conditions that rotation and displacements be single valued. These conditions are to be used in evaluating the constants of equation (19). The physical conditions of the problem will then be satisfied, for the stress function has been defined so as to satisfy the equilibrium conditions; the compatibility condition is satisfied by the use of biharmonic functions; the assumption of force-free boundaries is satisfied by conditions (18) and (19) on the stress function; and the constants a_{i1}, a_{i2}, and a_{i3} are to be evaluated so as to satisfy the conditions of single-valued rotation and displacements. That adjustment of the values of a_{i1}, a_{i2}, and a_{i3} is sufficient to satisfy the conditions of single-valued rotation and displacements will become evident in the next section. A method of determining the constants that will be appropriate for numerical techniques is needed.

DETERMINATION OF BOUNDARY CONSTANTS BY FORMATION OF SPECIAL SOLUTIONS

A method of using special solutions to determine a_{i1}, a_{i2}, and a_{i3} for a doubly connected domain at uniform temperature was described by Prager (reference 6). A similar method was suggested for the multiply connected domain by Southwell (reference 10). The method of Prager is here extended to domains with more than one hole and with temperature distributions present. The method is then modified to a form that is suitable for numerical techniques.

Let $\phi_{ij}(i=1, 2, \ldots k, \ldots n;\ j=1, 2, 3)$ be special solutions of equation (1) that are defined over the domain bounded by the external contour C_0 and the internal contours C_1, C_2, . . . C_k, . . . C_n. Because equation (1) is linear, the products of the ϕ_{ij} and the arbitrary constants a_{ij} may be superimposed to give the complete solution of the boundary-value problem according to the scheme

$$\phi=\sum_{i=1}^{n}\sum_{j=1}^{3}a_{ij}\phi_{ij} \tag{20}$$

provided that the boundary conditions for the ϕ_{ij} are properly selected and that the values of the boundary constants a_{ij} are properly evaluated.

The boundary conditions for the ϕ_{ij} must be selected so that the function given by equation (20) will satisfy equations (18) on all the boundaries. Because the stresses are given by second derivatives of ϕ, the addition of a linear function of the coordinates to ϕ will leave the stresses unaltered. The assumption is now made that this addition is accomplished so that the boundary constants of equations (18) and (19) are zero on the external boundary. The boundary conditions on the ϕ_{ij} for the external contour C_0 are therefore taken as

$$\phi_{ij}=\frac{\partial\phi_{ij}}{\partial n}=0 \text{ on } C_0$$

The boundary conditions for the ϕ_{ij} on the internal boundaries must now be selected so that $3n$ linearly independent solutions for the ϕ_{ij} will be obtained and so that the function defined by equation (20) will satisfy equations (18) on all the internal boundaries. This selection is accomplished as follows:

Let

$$\phi_{ij}=\frac{\partial \phi_{ij}}{\partial n}=0 \tag{21}$$

on all boundaries except C_k. On C_k, let

$$\phi_{k1}=x \qquad \frac{\partial \phi_{k1}}{\partial n}=\frac{dy}{ds} \tag{22}$$

$$\phi_{k2}=y \qquad \frac{\partial \phi_{k2}}{\partial n}=-\frac{dx}{ds} \tag{23}$$

$$\phi_{k3}=1 \qquad \frac{\partial \phi_{k3}}{\partial n}=0 \tag{24}$$

Certain geometrical aspects of the choices expressed by equations (21) to (24) are now mentioned. The observation is made, for example, that the equation

$$\frac{\partial \phi_{k1}}{\partial n}=\frac{dy}{ds}$$

is a restriction on ϕ_{k1} in addition to the restriction of $\phi_{k1}=x$ on C_k, inasmuch as $\phi_{k1}=x$ prescribes values of ϕ_{k1} only on the line defining the boundary C_k, whereas the normal derivative

$$\frac{\partial \phi_{k1}}{\partial n}=\frac{dy}{ds}$$

specifies the rate of change of ϕ_{k1} as the boundary is crossed in a direction normal to the boundary.

The slope of the plane $\phi_{k1}=x$ in a direction normal to the contour C_k is $\partial\phi_{k1}/\partial n$; but because $\phi_{k1}=x$ for this plane, the slope may be written as $\partial x/\partial n$. In the direction n, the slope of the surface ϕ_{k1} is given by equation (22) as

$$\frac{\partial \phi_{k1}}{\partial n}=\frac{dy}{ds}$$

but in the xy-plane,

$$\frac{dy}{ds}=\frac{\partial x}{\partial n}$$

Therefore, in the direction of n, the slope of the surface ϕ_{k1} (in equation (22)) has been taken equal to the slope of the plane $\phi_{k1}=x$ and because the intersection C'_k of the cylinder through C_k with the surface ϕ_{k1} lies in the plane $\phi_{k1}=x$, the surface ϕ_{k1} is tangent to the plane $\phi_{k1}=x$ along C'_k. The particular solution $a_{k1}\phi_{k1}$ therefore defines a surface tangent to the plane $\phi_{k1}=a_{k1}x$. Equations (23) require that ϕ_{k2} be tangent to the plane $\phi_{k2}=y$ along the intersection C''_k of the cylinder through C_k with the surface ϕ_{k2}, and equations (24) require that ϕ_{k3} be tangent to the plane $\phi_{k3}=1$ along the intersection C'''_k of the cylinder through C_k with the surface ϕ_{k3}.

The three special solutions, $a_{k1}\phi_{k1}$, $a_{k2}\phi_{k2}$, and $a_{k3}\phi_{k3}$, associated with C_k are seen to be tangent to three planes, one of which passes through the y-axis, one of which passes through the x-axis, and one of which is parallel to the xy-plane. Determination of a_{k1}, a_{k2}, and a_{k3} is seen to be equivalent to determining the slope of the plane through the y-axis, the slope of the plane through the x-axis, and the height of the plane parallel to the xy-plane. Superposition according to

$$\phi_k=a_{k1}\phi_{k1}+a_{k2}\phi_{k2}+a_{k3}\phi_{k3}$$

is thus seen to satisfy the requirements of equations (18) and (19), but the constants a_{k1}, a_{k2}, and a_{k3} must be properly chosen. Superposition of all the $a_{ij}\phi_{ij}$ will still leave equations (18) and (19) satisfied on C_k because of the requirements laid down on all the ϕ_{ij} by equation (21).

The ϕ_{ij} were defined as special solutions of equation (1), which is equivalent to writing

$$\nabla^4\phi_{ij}=0$$

$$(i=1, 2, \ldots k, \ldots n; j=1, 2, 3) \tag{25}$$

Equations (25) together with boundary-condition equations (21) to (24) constitute $3n$ boundary-value problems for the $3n$ particular solutions ϕ_{ij}. These individual boundary-value problems with ϕ_{ij} and $\partial\phi_{ij}/\partial n$ specified on every boundary are now to be solved by methods already described by Fox and Southwell (reference 7). With the assumption that the ϕ_{ij} are so evaluated, the next step is to calculate the values of a_{ij}.

The method for determining the a_{ij} may be symbolically expressed by substituting the complete integral as expressed by equation (20) into the contour integrals of equations (10), (16), and (17). This substitution requires that, on each kth integration path (enclosing the kth internal boundary), contour integrals involving all the ϕ_{ij} be formed as coefficients of the a_{ij}. Formation of these contour integrals then permits writing simultaneous equations for each kth boundary ($k=1, 2, \ldots n$) so that $3n$ equations are obtained for the a_{ij} where the contour integrals involving the ϕ_{ij} become coefficients of the a_{ij}:

$$\sum_{i=1}^{n}\sum_{j=1}^{3} a_{ij}\oint_{P_k}\frac{\partial\nabla^2\phi_{ij}}{\partial n}\,ds=-\frac{\alpha E}{1-\nu}\oint_{P_k}\frac{\partial T}{\partial n}\,ds \tag{26}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{3} a_{ij}\oint_{P_k}\left(y\,\frac{\partial\nabla^2\phi_{ij}}{\partial n}-x\,\frac{d\nabla^2\phi_{ij}}{ds}\right)ds =-\frac{\alpha E}{1-\nu}\oint_{P_k}\left(y\,\frac{\partial T}{\partial n}-x\,\frac{dT}{ds}\right)ds \tag{27}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{3} a_{ij}\oint_{P_k}\left(y\,\frac{d\nabla^2\phi_{ij}}{ds}+x\,\frac{\partial\nabla^2\phi_{ij}}{\partial n}\right)ds =-\frac{\alpha E}{1-\nu}\oint_{P_k}\left(y\,\frac{dT}{ds}+x\,\frac{\partial T}{\partial n}\right)ds \tag{28}$$

Because there are exactly $3n$ equations with exactly $3n$ unknowns (the a_{ij}), the determination of these constants is sufficient to insure the occurrence of single-valued rotation and displacements.

NUMERICAL TECHNIQUE

The purposes of this section are (1) to illustrate, by a concrete example, the detailed steps by which a solution of a problem may be obtained, (2) to show, by comparison with a problem for which the solution is already known, that the application of relaxation procedures to the finite-difference method outlined in the preceding part of this report yields a method giving results that approximate the correct answers, and (3) to present the results of an investigation of some factors affecting the accuracy of the answers for some particular conditions encountered in the relaxation solution of a thermal stress problem.

ILLUSTRATIVE EXAMPLE

The detailed instructions enabling the relaxation calculation of thermal stresses in an irregular cylinder are presented in appendix C. The particular problem illustrated is that of a concentric circular cylinder possessing an asymmetrical temperature distribution; however, the description of method may be applied to cylinders of more complex shape. The choice of the concentric circular cylinder enabled the relaxation work for the stress functions to be confined to a 90° sector. There would be no fundamental distinction in carrying out the calculations for an irregular profile—relaxation would merely have to be performed over the entire cross section entailing more labor. (Although circular boundaries were involved, the advantages of using polar coordinates were not utilized. The use of rectangular coordinates in the presence of circular boundaries involves boundary technique problems typical of a more irregular region.)

The results of the relaxation calculation (reference 11) according to Laplace's equation for the temperature distribution are presented in figure 1. Results of relaxation calculations for special solutions of the biharmonic equation are presented in figures 2 to 4. Contour integrals were calculated as illustrated by table I and the stress function is presented in table II. Tangential stresses (table II) were calculated by computing the second derivatives of the stress function with respect to radius (reference 8, p. 53) according to the five-point method of reference 12 described in appendix D.

Exact values of radial and tangential stress were calculated as indicated in appendix E and are also listed in table II. The maximum tangential stress is seen to be much larger than the maximum radial stress. The error in the relaxation calculation of the maximum tangential stress was about 5 percent. Comparison of the values of tangential stress calculated by the exact method with values calculated by the relaxation method is also presented in figure 5.

INVESTIGATION OF FACTORS AFFECTING ACCURACY

Several factors influencing the accuracy obtainable in calculating thermal stresses by the method just described were investigated by applying various calculation techniques to a problem for which answers could be calculated by exact mathematical methods. The example chosen consisted of a concentric cylinder subjected to a symmetrical temperature distribution. Details of the relaxation calculation are presented in appendix F.

The results of the relaxation calculation for the temperature distribution are presented in figure 6. Because dimensions of the cylinder were exactly the same as those of the illustrative example previously discussed, the special solutions ϕ_{11}, ϕ_{12}, and ϕ_{13} determined in that example are usable in the present example. This situation illustrates an important feature of the use of relaxation methods in calculating thermal stresses; that is, once the time consuming biharmonic relaxation work for the special functions ϕ_{11}, ϕ_{12}, and ϕ_{13} has been completed for a given shape of body, relatively little extra work is required to study the effects on thermal stresses of changes in temperature distribution.

For the symmetrical temperature distribution of figure 6, the boundary constants a_{11} and a_{12} were found to vanish. The values of a_{13} for the paths a, b, and c of figure 4 are presented in table III. The Airy stress function and the stresses calculated from it by numerical differentiation according to appendix D are listed in table IV.

Exact values of Airy's stress function were calculated according to appendix G. The value of the arbitrary constant D in equation (G4) was adjusted to give $\phi=0$ for $r=12$. Results are listed in table IV. Second derivatives of the exact values of Airy's stress function were then calculated by the numerical methods of appendix D to give the tangential stress values listed in table IV. Exact values of tangential stress were calculated using the second of equations (E1) and are also listed in table IV.

Comparison of the errors in tangential stress of table IV shows that the errors associated with the numerical differentiation of the relaxation-calculated Airy function were much larger than those associated with the numerical differentiation of the exact Airy function. (Errors in the maximum stress were 21.5 and 6.5 percent, respectively.) This comparison suggests that the relaxation-calculated Airy function was an important source of error.

The relaxation-calculated stress function was calculated as the product $a_{13}\phi_{13}$ (appendix F). An exact value of a_{13} was calculated by observing that the exact values of ϕ in table IV range from zero at $r=12$ to $-467,627$ at $r=4$; whereas in figure 4 the values of ϕ_{13} range from zero at $r=12$ to 1000 at $r=4$. The exact value of a_{13} is therefore

$$a_{13}=\frac{-467,627}{1000}=-467.627$$

As indicated by the data of table III, the errors associated with the processes of integrating difference quotients to calculate the boundary constants can be significant but are small if the calculated values of the boundary constants are averaged over several paths.

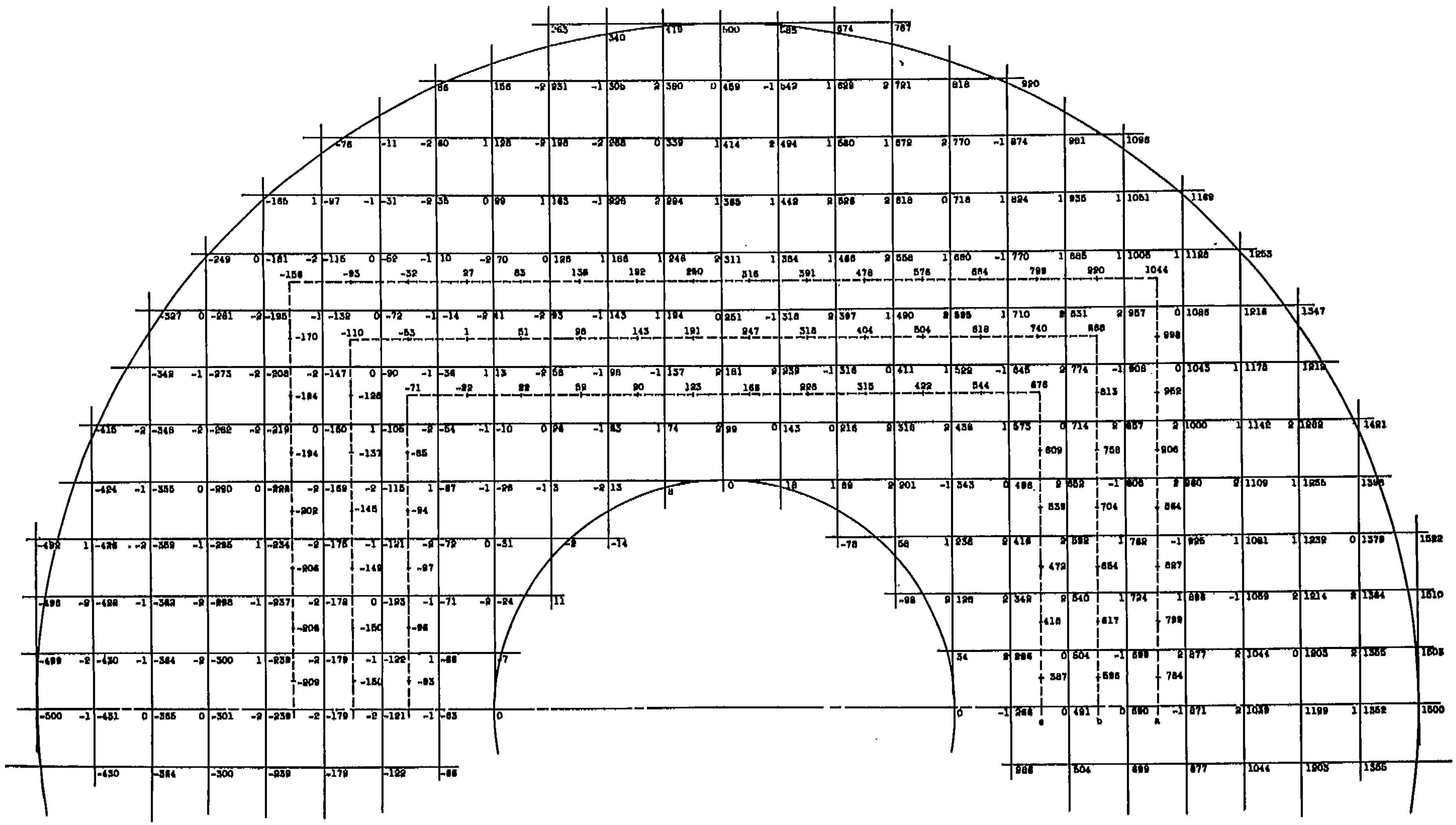

FIGURE 1.—Results of relaxation calculation of asymmetrical temperature distribution. Temperatures in degrees Fahrenheit: on inner boundary, $T=0$; on outer boundary, $T=500+1000\cos\theta$. Temperature values on integration paths are obtained by averaging temperature values at four nearest nodal points.

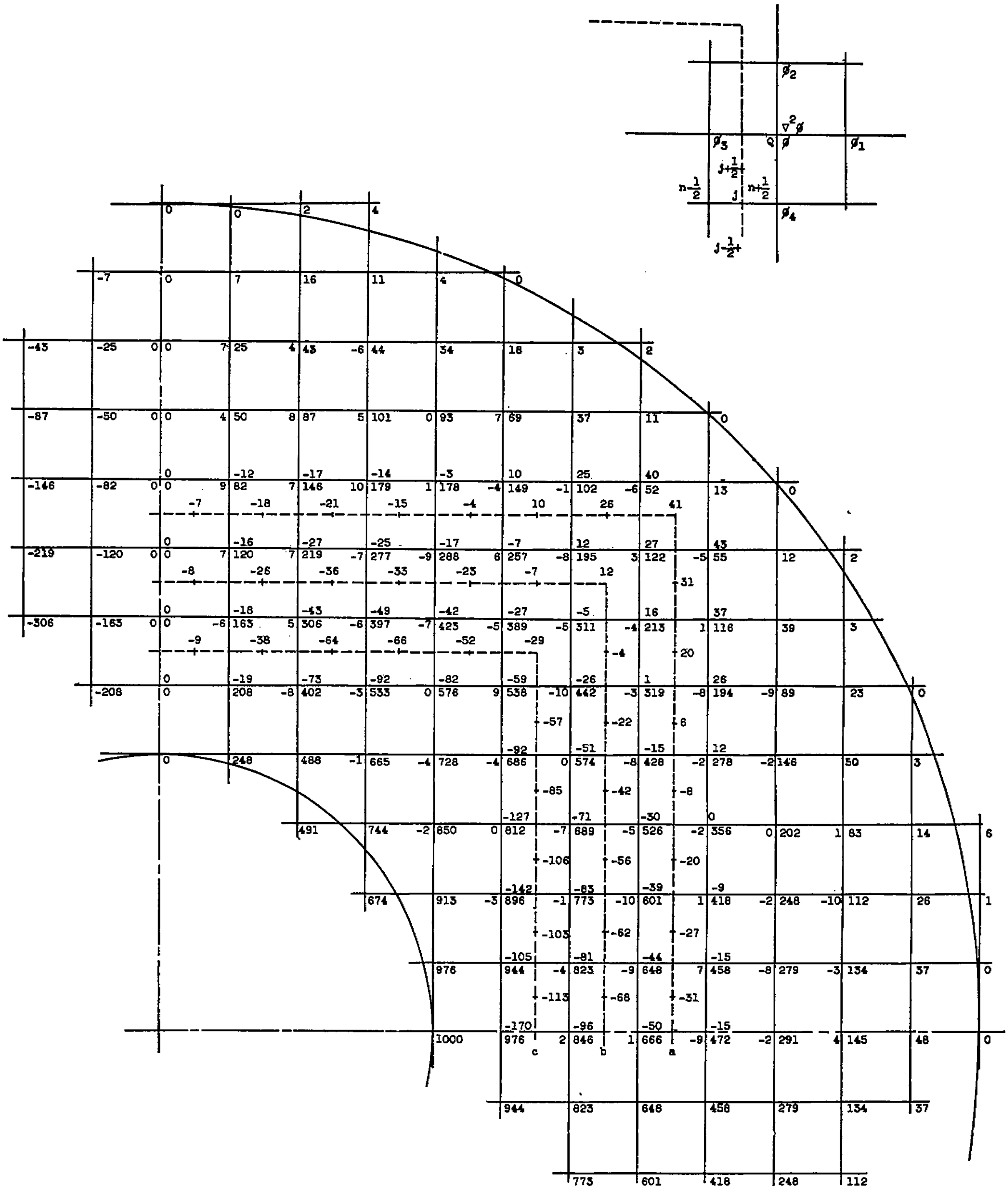

FIGURE 2.—Results of relaxation calculation of ϕ_{11}. Values of $\nabla^2\phi$ on integration paths are obtained by averaging $\nabla^2\phi$ values at four nearest nodal points.

$$\nabla^2\phi = \phi_1 + \phi_2 + \phi_3 + \phi_4 - 4\phi$$

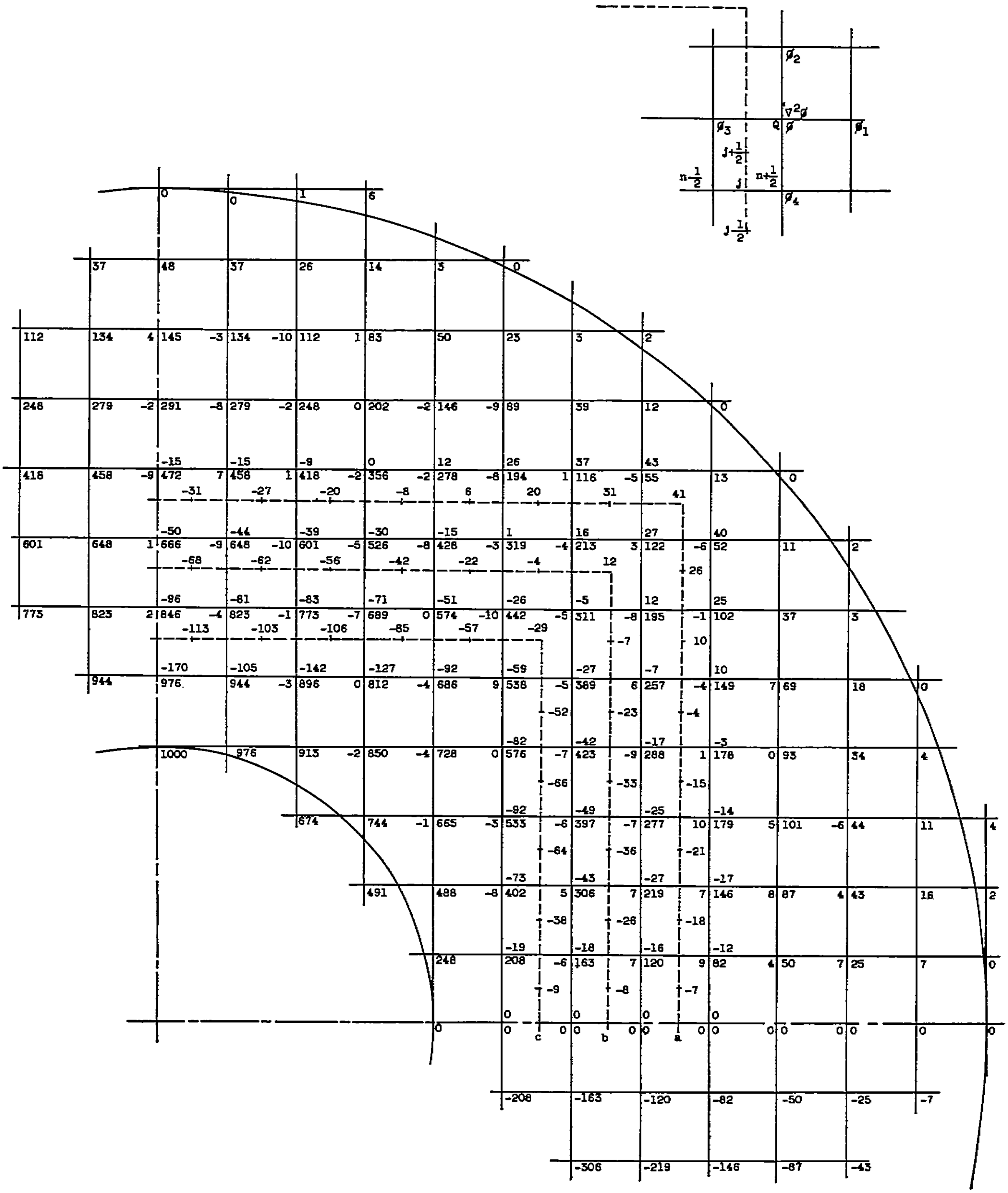

FIGURE 3.—Results of relaxation calculation of ϕ_{II}. Values of $\nabla^2\phi$ on integration paths are obtained by averaging $\nabla^2\phi$ values at four nearest nodal points.

$$\nabla^2\phi=\phi_1+\phi_2+\phi_3+\phi_4-4\phi$$

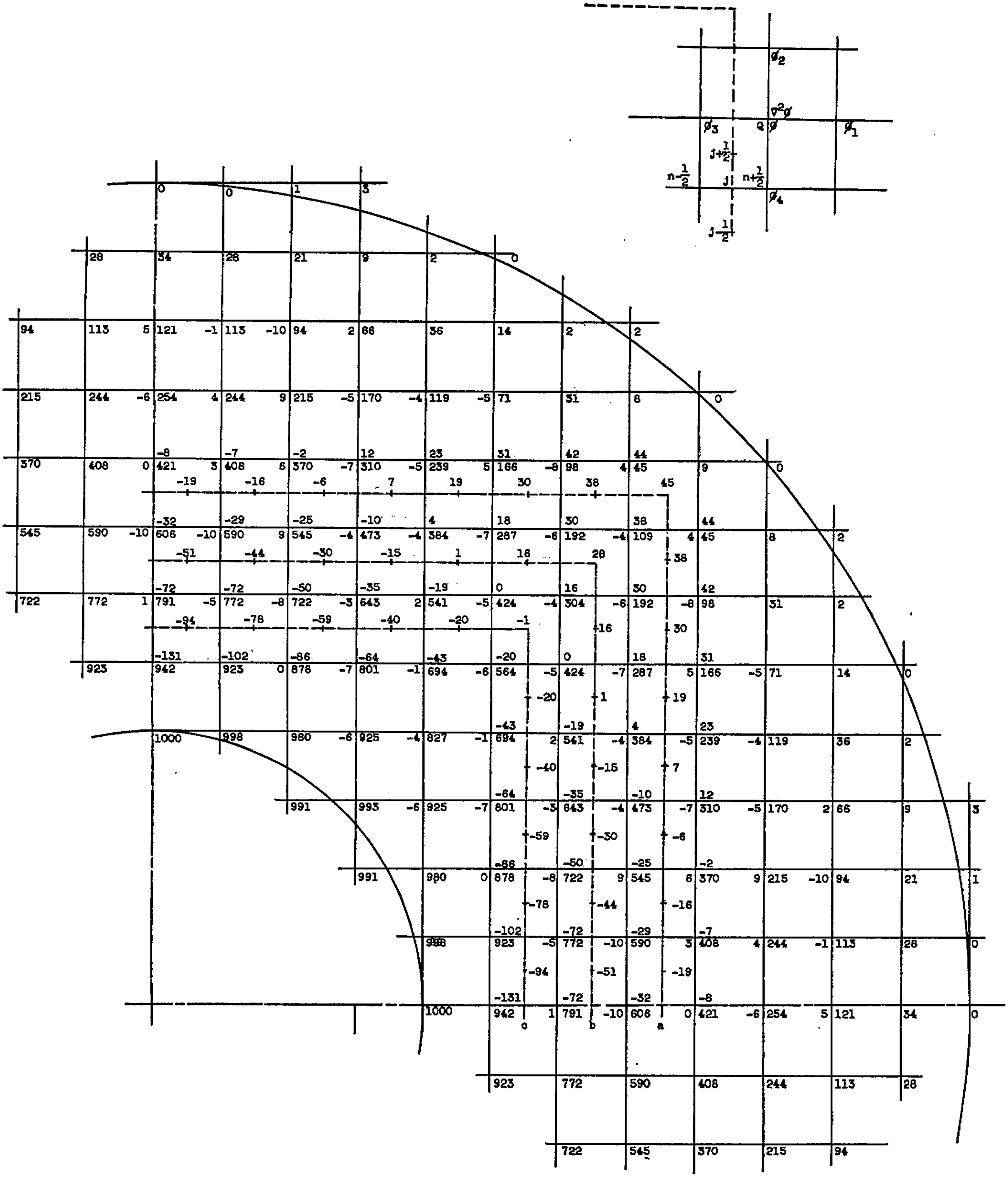

FIGURE 4.—Results of relaxation calculation of ϕ_{11}. Values of $\nabla^2\phi$ on integration paths are obtained by averaging $\nabla^2\phi$ values at four nearest nodal points.

$$\nabla^2\phi = \phi_1 + \phi_2 + \phi_3 + \phi_4 - 4\phi$$

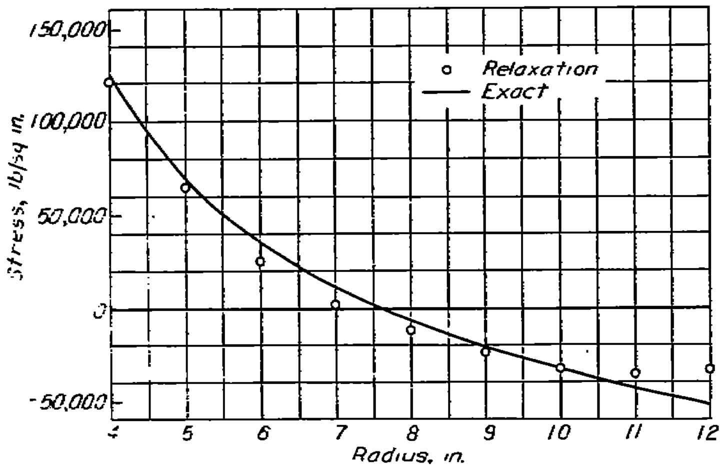

FIGURE 5.—Comparison of tangential stresses calculated by relaxation method with exact stresses; concentric cylinder with asymmetrical temperature distribution.

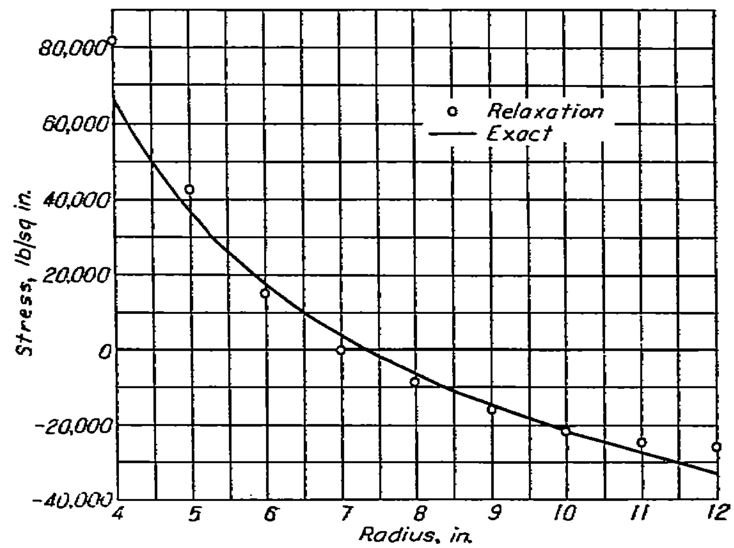

FIGURE 7.—Comparison of tangential stresses calculated by relaxation method with exact stresses; concentric cylinder with symmetrical temperature distribution.

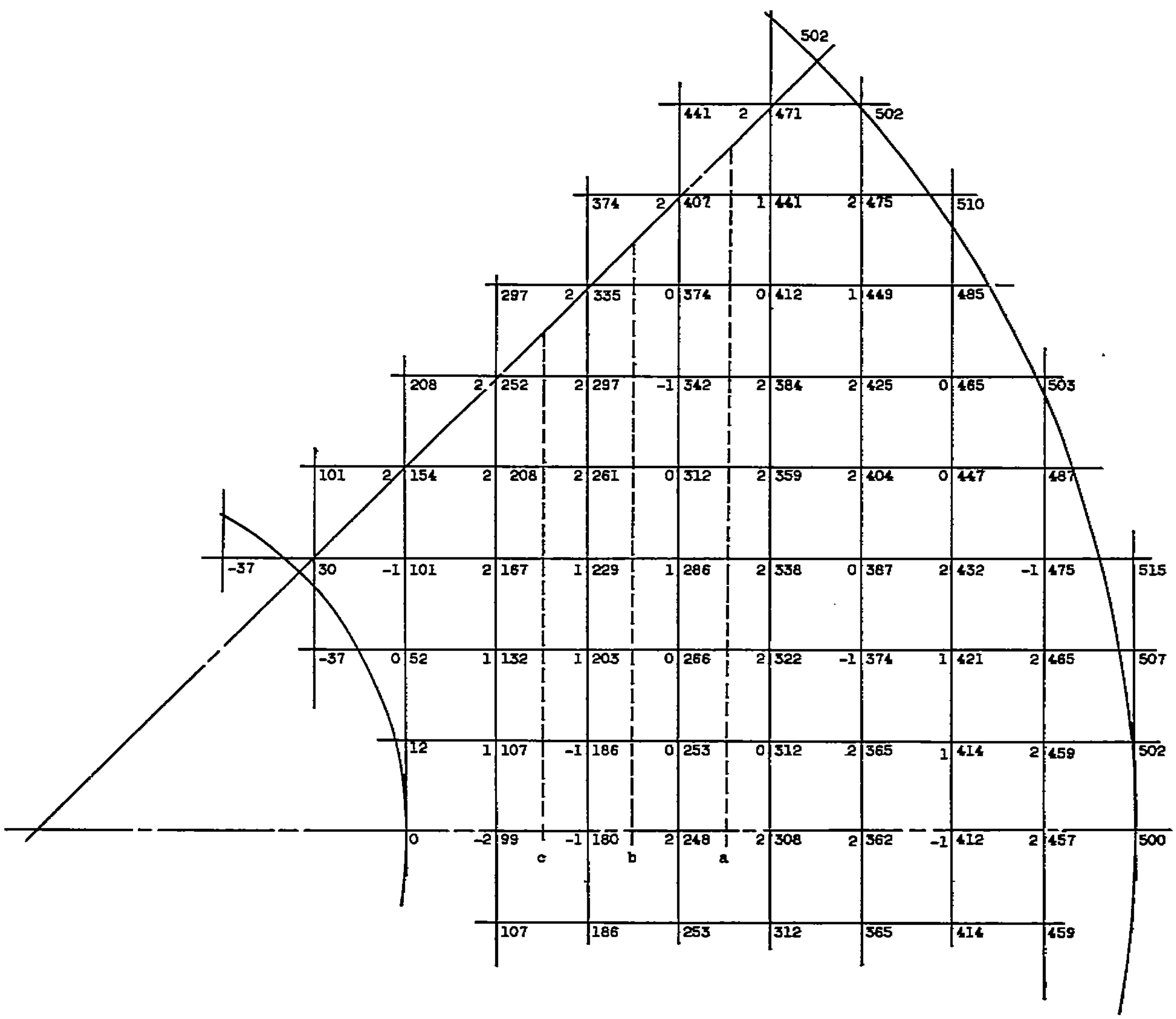

FIGURE 6.—Results of relaxation calculation for temperature distribution in concentric hollow cylinder with symmetrical temperature distribution. Inner radius, 4 inches; outer radius, 12 inches; inner surface, 0° F; outer surface, 500° F.

The extent to which errors in the relaxation calculation of the Airy function influence errors in the stresses was evaluated by rounding off the previously determined exact values of Airy's stress function to three significant figures. Corresponding stresses and errors are presented in table IV. In general, the errors associated with the three-figure Airy function are seen to be significantly lower than those associated with the relaxation-calculated Airy function. Apparently, the equivalent of three-figure accuracy was not achieved in the relaxation calculation of the Airy function. Comparison of the values of tangential stress calculated by the exact method with values calculated by the relaxation method is also presented in figure 7. Improved accuracy could be accomplished by (1) further reduction of residuals with the introduction of another significant figure, (2) the use of a finer net spacing, (3) the use of more elegant finite-difference methods, or (4) some combination of (1), (2), and (3). A critical discussion of some factors influencing the accuracy of relaxation procedures is contained in reference 13.

CONCLUDING REMARKS

This investigation has yielded a numerical method for calculating thermal stress in a cooled irregular cylinder possessing one or more cooling passages under steady-state temperature conditions. Application of the method to structures such as internally cooled turbine blades is suggested. The use of relaxation methods and elementary methods of finite differences has been found to give approximations to correct values when compared with previously known solutions for concentric circular cylinders possessing symmetrical and asymmetrical temperature distributions.

LEWIS FLIGHT PROPULSION LABORATORY
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS
CLEVELAND, OHIO, *May 10, 1951*

APPENDIX A

DERIVATION OF BIHARMONIC EQUATION

An outline of the derivation of the governing partial differential equation for the stress function is given in reference 4. A detailed derivation using the conditions imposed in the section on assumptions follows:

The defining equations for the normal strains (reference 8, p. 7) are

$$\left.\begin{aligned}\epsilon_x&=\frac{\partial u}{\partial x}\\ \epsilon_y&=\frac{\partial v}{\partial y}\\ \epsilon_z&=\frac{\partial w}{\partial z}\end{aligned}\right\}\tag{A1}$$

and the defining equations for the shearing strains are

$$\left.\begin{aligned}\gamma_{xy}&=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\\ \gamma_{xz}&=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\\ \gamma_{yz}&=\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\end{aligned}\right\}\tag{A2}$$

The defining equation for the rotation (reference 8, p. 162) is

$$\omega=\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\tag{A3}$$

The assumption that the strains and rotation are constant in the direction of the z-axis (axis of the cylinder) permits writing

$$\frac{\partial \epsilon_x}{\partial z}=\frac{\partial \epsilon_y}{\partial z}=\frac{\partial \epsilon_z}{\partial z}=\frac{\partial \gamma_{xy}}{\partial z}=\frac{\partial \gamma_{xz}}{\partial z}=\frac{\partial \gamma_{yz}}{\partial z}=\frac{\partial \omega}{\partial z}=0\tag{A4}$$

The conditions that the strains be compatible with displacements specified by u, v, and w are (reference 8, p. 196)

$$\left.\begin{aligned}\frac{\partial^2\epsilon_x}{\partial y^2}+\frac{\partial^2\epsilon_y}{\partial x^2}&=\frac{\partial^2\gamma_{xy}}{\partial x\partial y}\\ \frac{\partial^2\epsilon_y}{\partial z^2}+\frac{\partial^2\epsilon_z}{\partial y^2}&=\frac{\partial^2\gamma_{yz}}{\partial y\partial z}\\ \frac{\partial^2\epsilon_z}{\partial x^2}+\frac{\partial^2\epsilon_x}{\partial z^2}&=\frac{\partial^2\gamma_{xz}}{\partial x\partial z}\\ \frac{\partial^2\epsilon_x}{\partial y\partial z}&=\frac{1}{2}\frac{\partial}{\partial x}\left(-\frac{\partial\gamma_{yz}}{\partial x}+\frac{\partial\gamma_{xz}}{\partial y}+\frac{\partial\gamma_{xy}}{\partial z}\right)\\ \frac{\partial^2\epsilon_y}{\partial x\partial z}&=\frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial\gamma_{yz}}{\partial x}-\frac{\partial\gamma_{xz}}{\partial y}+\frac{\partial\gamma_{xy}}{\partial z}\right)\\ \frac{\partial^2\epsilon_z}{\partial x\partial y}&=\frac{1}{2}\frac{\partial}{\partial z}\left(\frac{\partial\gamma_{yz}}{\partial x}+\frac{\partial\gamma_{xz}}{\partial y}-\frac{\partial\gamma_{xy}}{\partial z}\right)\end{aligned}\right\}\tag{A5}$$

From equations (A3) and (A4), equations (A5) become

$$\left.\begin{aligned}\frac{\partial^2\epsilon_x}{\partial y^2}+\frac{\partial^2\epsilon_y}{\partial x^2}&=\frac{\partial^2\gamma_{xy}}{\partial x\partial y}\\ \frac{\partial^2\epsilon_z}{\partial y^2}&=0\\ \frac{\partial^2\epsilon_z}{\partial x^2}&=0\\ \frac{\partial^2\epsilon_x}{\partial y\partial z}&\equiv 0\\ \frac{\partial^2\epsilon_y}{\partial x\partial z}&\equiv 0\\ \frac{\partial^2\epsilon_z}{\partial x\partial y}&=0\end{aligned}\right\}\tag{A6}$$

The first of equations (A6) remains, whereas the fourth and fifth of equations (A6) vanish identically. The second, third, and sixth of equations (A6) may be written

$$\frac{\partial^3 w}{\partial y^2 \partial z}=\frac{\partial^3 w}{\partial x^2 \partial z}=\frac{\partial^3 w}{\partial x \partial y \partial z}=0 \tag{A7}$$

Because of the assumption that plane sections initially normal to the axis of the cylinder remain plane after the deformation, the displacement in the z-direction may be written

$$w=f(z)x+g(z)y+h(z)$$

showing that equations (A7) vanish identically. The preceding proof that the last five of equations (A6) vanish identically is essentially a demonstration that these compatibility conditions are satisfied as a consequence of the initial assumptions. Of equations (A6), the only nonvanishing equation is now the compatibility condition

$$\frac{\partial^2 \epsilon_x}{\partial y^2}+\frac{\partial^2 \epsilon_y}{\partial x^2}=\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \tag{A8}$$

The generalized Hooke's law equations (reference 8, p. 204) are

$$\left.\begin{aligned}\epsilon_x&=\frac{1}{E}\,[\sigma_x-\nu(\sigma_y+\sigma_z)]+\alpha T\\ \epsilon_y&=\frac{1}{E}\,[\sigma_y-\nu(\sigma_x+\sigma_z)]+\alpha T\\ \epsilon_z&=\frac{1}{E}\,[\sigma_z-\nu(\sigma_x+\sigma_y)]+\alpha T\end{aligned}\right\} \tag{A9}$$

Elimination of σ_z between these equations yields

$$\left.\begin{aligned}\epsilon_x&=\frac{1}{E}\,[(1-\nu^2)\sigma_x-\nu(1+\nu)\sigma_y+(1+\nu)\alpha ET-\nu E\epsilon_z]\\ \epsilon_y&=\frac{1}{E}\,[(1-\nu^2)\sigma_y-\nu(1+\nu)\sigma_x+(1+\nu)\alpha ET-\nu E\epsilon_z]\end{aligned}\right\} \tag{A10}$$

Airy's stress function is defined by

$$\left.\begin{aligned}\sigma_x&=\frac{\partial^2\phi}{\partial y^2}\\ \sigma_y&=\frac{\partial^2\phi}{\partial x^2}\\ \tau_{xy}&=-\frac{\partial^2\phi}{\partial x\partial y}\end{aligned}\right\} \tag{2}$$

Equations (A10) and (2) are used with the direct method of calculating thermal stresses; that is, the equilibrium, boundary, and compatibility conditions are used without regarding temperature terms as body and surface forces. The equilibrium equations (reference 8, p. 195) are therefore written without body forces. These equations were reduced to those of reference 8, page 21, by using the assumption that plane sections initially normal to the axis of the cylinder remain plane after the deformation, so that

$$\left.\begin{aligned}\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}&=0\\ \frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{xy}}{\partial x}&=0\end{aligned}\right\} \tag{A11}$$

Substitution of expressions (2) in equations (A11) shows that the stress function has been defined so as to satisfy the equilibrium equations identically. The condition remaining to be satisfied by ϕ is the compatibility condition (A8). Substitution of expressions (2) in equations (A10) yields

$$\left.\begin{aligned}\epsilon_x&=\frac{1}{E}\left(\frac{\partial^2\phi}{\partial y^2}-\nu\,\frac{\partial^2\phi}{\partial x^2}-\nu^2\nabla^2\phi-\nu E\epsilon_z\right)+(1+\nu)\alpha T\\ \epsilon_y&=\frac{1}{E}\left(\frac{\partial^2\phi}{\partial x^2}-\nu\,\frac{\partial^2\phi}{\partial y^2}-\nu^2\nabla^2\phi-\nu E\epsilon_z\right)+(1+\nu)\alpha T\end{aligned}\right\} \tag{A12}$$

The shearing strain is expressed (reference 8, p. 10) in terms of the shearing stress by

$$\begin{aligned}\gamma_{xy}&=\frac{2(1+\nu)}{E}\,\tau_{xy}\\ &=-\frac{2(1+\nu)}{E}\,\frac{\partial^2\phi}{\partial x\partial y}\end{aligned} \tag{A13}$$

By use of equations (A6), (A12), and (A13), equation (A8) can be written

$$\nabla^4\phi=-\frac{\alpha E}{1-\nu}\,\nabla^2 T$$

For the assumed temperature conditions,

$$\nabla^2 T=0$$

and hence the governing equation for ϕ is

$$\nabla^4\phi=0 \tag{1}$$

APPENDIX B

CALCULATION OF AXIAL STRESS

After σ_x and σ_y have been calculated according to equation (2), the normal stress σ_z in the axial direction may be determined. The third of equations (A9) is

$$\epsilon_z=\frac{1}{E}\,[\sigma_z-\nu(\sigma_x+\sigma_y)]+\alpha T$$

from which

$$\sigma_z=\epsilon_z E+\nu(\sigma_x+\sigma_y)-\alpha ET \tag{B1}$$

As stated in the section ASSUMPTIONS, the cylinder is free from applied forces and therefore the total force in the axial direction and the x- and y-components of a bending couple are zero.

$$\left.\begin{aligned}\iint_A \sigma_z\,dx\,dy&=0\\ \iint_A x\,\sigma_z\,dx\,dy&=0\\ \iint_A y\,\sigma_z\,dx\,dy&=0\end{aligned}\right\} \tag{B2}$$

Because assumptions numbers 6 and 7 in the section ASSUMPTIONS, the strain in the z-direction may be written

$$\epsilon_z=ax+by+c \tag{B3}$$

where a, b, and c are constants.

Substitution of expressions given by equations (B1) and (B3) in equations (B2) gives

$$\left.\begin{aligned}&a\iint_A x\,dx\,dy+b\iint_A y\,dx\,dy+c\iint_A dx\,dy\\ &\qquad=-\frac{1}{E}\iint_A[\nu(\sigma_x+\sigma_y)-\alpha ET]dx\,dy\\ &a\iint_A x^2dx\,dy+b\iint_A x\,y\,dx\,dy+c\iint_A x\,dx\,dy\\ &\qquad=-\frac{1}{E}\iint_A x[\nu(\sigma_x+\sigma_y)-\alpha ET]dx\,dy\\ &a\iint_A x\,y\,dx\,dy+b\iint_A y^2dx\,dy+c\iint_A y\,dx\,dy\\ &\qquad=-\frac{1}{E}\iint_A y[\nu(\sigma_x+\sigma_y)-\alpha ET]dx\,dy\end{aligned}\right\} \tag{B4}$$

Equations (B4) are three simultaneous equations from which the constants a, b, and c may be evaluated. Substitution of the expression for ϵ_z given in equation (B3) for ϵ_z in equation (B1) gives

$$\sigma_z=E(ax+by+c)+\nu(\sigma_x+\sigma_y)-\alpha ET \tag{3}$$

from which the axial stresses may be determined.

If the origin of the coordinate axes is taken at the center of gravity of the cross section,

$$\iint_A x\,dx\,dy=\iint_A y\,dx\,dy=0 \tag{B5}$$

and in any case,

$$\iint_A dx\,dy=A \tag{B6}$$

where A is the area of the cross section.

Where sufficient symmetry is present, the orientation of the principal axes of inertia can often be determined by inspection. A method of determining the principal axes and principal moments of inertia for arbitrarily shaped areas is presented in reference 14. If the orientation of the coordinate axes is chosen so it coincides with that of the principal axes, then

$$\iint_A y^2\,dx\,dy=I_x \tag{B7}$$

$$\iint_A xy\,dx\,dy=0 \tag{B8}$$

$$\iint_A x^2\,dx\,dy=I_y \tag{B9}$$

where I_x is the moment of inertia about the x-axis, and I_y is the moment of inertia about the y-axis. Simplification of equations (B4) using equations (B5), (B6), (B7), (B8), and (B9) furnishes explicit solutions for a, b, and c as follows:

$$\left.\begin{aligned}a&=-\frac{1}{EI_y}\iint_A x[\nu(\sigma_x+\sigma_y)-\alpha ET]dx\,dy\\ b&=-\frac{1}{EI_x}\iint_A y[\nu(\sigma_x+\sigma_y)-\alpha ET]dx\,dy\\ c&=-\frac{1}{AE}\iint_A[\nu(\sigma_x+\sigma_y)-\alpha ET]dx\,dy\end{aligned}\right\} \tag{B10}$$

Substitution of equations (10) in equation (3) gives

$$\begin{aligned}\sigma_z=\nu(\sigma_x+\sigma_y)-\alpha ET-&\\ \frac{x}{I_y}\iint_A x[\nu(\sigma_x+\sigma_y)-\alpha ET]dx\,dy-&\\ \frac{y}{I_x}\iint_A y[\nu(\sigma_x+\sigma_y)-\alpha ET]dx\,dy-&\\ \frac{1}{A}\iint_A[\nu(\sigma_x+\sigma_y)-\alpha ET]dx\,dy&\end{aligned} \tag{B11}$$

APPENDIX C

DETAILS OF RELAXATION CALCULATION OF TANGENTIAL STRESS IN CONCENTRIC CYLINDER WITH ASYMMETRIC TEMPERATURE DISTRIBUTION

The temperature distribution was assumed to be determined by temperatures in degrees Fahrenheit of zero on the internal boundary of 4-inch radius and $500+1000\cos\theta$ on the external boundary of 12-inch radius. The values of the elastic constants were assumed to be

$\alpha=8.0\times10^{-6}$ (in./in./°F)

$E=17.5\times10^{6}$ (lb/sq in.)

$\nu=0.3$

The results of the relaxation calculation for the temperature distribution according to the technique described in reference 11 is presented in figure 1. In all calculations, the origin of coordinates was located at the center of the cylinder. In the present case (one internal boundary), the indices of equations (21), (22), (23), and (24) become

$$i=k=n=1$$

$$j=1, 2, 3$$

In order to eliminate the use of decimals in the work of relaxation, equations (21), (22), (23), and (24) were modified as follows:

On the exterior boundary,

$$\phi_{11}=\phi_{12}=\phi_{13}=\frac{\partial\phi_{11}}{\partial n}=\frac{\partial\phi_{12}}{\partial n}=\frac{\partial\phi_{13}}{\partial n}=0$$

On the interior boundary,

$$\phi_{11}=250\,x \qquad \frac{\partial\phi_{11}}{\partial n}=250\,\frac{dy}{ds}$$

$$\phi_{12}=250\,y \qquad \frac{\partial\phi_{12}}{\partial n}=-250\,\frac{dx}{ds}$$

$$\phi_{13}=1000 \qquad \frac{\partial\phi_{13}}{\partial n}=0$$

The boundary-value problems just defined for the biharmonic functions ϕ_{11}, ϕ_{12}, and ϕ_{13} were then solved by the techniques of references 7 and 11 to yield the solutions presented in figures 2 to 4. (The dimensions of the cylinder were such that the distance between nodal points could be conveniently taken as 1 inch. For more general cases a method of handling dimensions, nodal distances, and derivatives, such as described in reference 7, can be followed.)

The next step is to calculate the values of the constants a_{11}, a_{12}, and a_{13} according to equations (26), (27), and (28). To that end, the appropriate contour integrals were calculated as illustrated for functions involving ϕ_{11} and T along path c in table I. The numbering system for points in the tables corresponds to the numbering system exhibited by the small diagram in figure 2, where j is the station along the path of integration and $j=1$ on the positive x-axis.

With the use of averages of contour integrals for paths a, b, and c, the simultaneous equations (26), (27), and (28) become

$$1241.3\;a_{13}=-200\times2853$$

$$8466.3\;a_{12}=0$$

$$8466.3\;a_{11}=-200\times9532.3$$

from which

$$a_{13}=-459.7$$

$$a_{12}=0$$

$$a_{11}=-225.2$$

According to equation (20),

$$\phi=a_{11}\,\phi_{11}+a_{12}\,\phi_{12}+a_{13}\,\phi_{13}$$

from which

$$\phi=-225.2\,\phi_{11}-459.7\,\phi_{13}$$

The solution of this equation for points along the positive x-axis ($\theta=0°$) is presented in table II. Tangential stresses were calculated from the relaxation solution for the stress function by taking second derivatives with respect to radius according to the formulas of appendix D. These results are also presented in table II.

APPENDIX D

NUMERICAL DIFFERENTIATION OF AIRY'S STRESS FUNCTION

The purpose of this section is to present the methods used in approximating to the second derivatives of Airy's stress function. The function to be differentiated is expressed by

$$y=f(x)$$

where in the present application y is Airy's stress function and x is radial distance on a cross section of the hollow concentric cylinder. Where p is an integer, h is a uniform tabular interval of the independent variable, and x_0 is an arbitrary point from which the distance ph is measured, let

$$x_p=x_0+ph \tag{D1}$$

The values of second derivatives were calculated by the five-point formulas of reference 12. For the various values of p, these formulas are: With $p=0$,

$$D^2y_0=\frac{1}{12h^2}(35y_0-104y_1+114y_2-56y_3+11y_4) \tag{D2}$$

with $p=1$,

$$D^2y_1=\frac{1}{12h^2}(11y_0-20y_1+6y_2+4y_3-y_4) \tag{D3}$$

with $p=2$,

$$D^2y_2=\frac{1}{12h^2}(-y_0+16y_1-30y_2+16y_3-y_4) \tag{D4}$$

with $p=3$,

$$D^2y_3=\frac{1}{12h^2}(-y_0+4y_1+6y_2-20y_3+11y_4) \tag{D5}$$

with $p=4$,

$$D^2y_4=\frac{1}{12h^2}(11y_0-56y_1+114y_2-104y_3+35y_4) \tag{D6}$$

Equation (D2) was used to calculate derivatives at $r=4$, equation (D3) was used for $r=5$, equation (D4) was used for $r=6$ to 10, equation (D5) was used for $r=11$, and equation (D6) was used for $r=12$.

APPENDIX E

EXACT DETERMINATION OF STRESSES FOR CONCENTRIC CYLINDER WITH ASYMMETRIC TEMPERATURE DISTRIBUTION

The radial and tangential stresses, respectively, in the case of a symmetrical temperature distribution are given in reference 8 on page 372 by

$$\left.\begin{aligned}\sigma_r&=\frac{\alpha E T_t}{2(1-\nu)\ln\frac{b}{a}}\left[-\ln\frac{b}{r}-\frac{a^2}{b^2-a^2}\left(1-\frac{b^2}{r^2}\right)\ln\frac{b}{a}\right]\\ \sigma_\theta&=\frac{\alpha E T_t}{2(1-\nu)\ln\frac{b}{a}}\left[1-\ln\frac{b}{r}-\frac{a^2}{b^2-a^2}\left(1+\frac{b^2}{r^2}\right)\ln\frac{b}{a}\right]\end{aligned}\right\}\tag{E1}$$

where a is the radius of the internal boundary and b is the radius of the external boundary, and where T_t is the excess of the inner boundary temperature over the temperature of the external boundary.

From reference 4, the stresses in a concentric circular cylinder possessing the temperature distribution

$$T=A\ \ln\ r+A_0+\sum_{n=1}^{\infty}(A_nr^n+C_nr^{-n})\cos n\theta+(-B_nr^n+D_nr^{-n})\sin n\theta \tag{E2}$$

where A, B, C, and D are constants, are

$$\left.\begin{aligned}\sigma_r=&\frac{\alpha EA}{2(1-\nu)}\left[\ln\frac{b}{r}-\frac{a^2}{r^2}\frac{b^2-r^2}{b^2-a^2}\ln\frac{b}{a}\right]-\\ &\frac{\alpha E}{2(1-\nu)}\left[\frac{(b^2-r^2)(r^2-a^2)}{(a^2+b^2)r^3}\right](C_1\cos\theta+D_1\sin\theta)\\ \sigma_\theta=&\frac{\alpha EA}{2(1-\nu)}\left[\ln\frac{b}{r}+\frac{a^2}{r^2}\frac{b^2+r^2}{b^2-a^2}\ln\frac{b}{a}-1\right]-\\ &\frac{\alpha E}{2(1-\nu)}\left[\frac{4a^2r^2-(3r^2-b^2)(r^2+a^2)}{(a^2+b^2)r^3}\right](C_1\cos\theta+D_1\sin\theta)\end{aligned}\right\}\tag{E3}$$

The temperature distribution in °F is specified by $T=0$ on the internal boundary and by $T=500+1000\cos\theta$ on the external boundary. The stresses will be calculated by superposing stresses for a symmetrical temperature distribution on those calculated for a special asymmetrical distribution. The given temperature distribution is resolved into two components: a symmetrical component defined by $T=0$ on the interior boundary and $T=500$ on the exterior boundary; and the special asymmetrical component defined by boundary temperatures of $T=0$ on the interior boundary and $T=1000\cos\theta$ on the exterior boundary.

The stresses calculated for the symmetrical component of the temperature distribution were calculated according to equations (E1).

Now equation (E2) is to be written so as to satisfy the boundary conditions on the asymmetrical component of the temperature distribution. The boundary conditions are

$$\begin{aligned}&\text{at } r=a & T&=0\\ &\text{at } r=b & T&=1000\cos\theta\end{aligned}$$

In order to satisfy these conditions and equation (E2), let

$$n=1$$

$$A=A_0=B_n=D_n=0 \tag{E4}$$

Equation (E2) is thereby reduced to

$$T=\left(A_1r+\frac{C_1}{r}\right)\cos\theta$$

On $r=a$,

$$0=\left(A_1a+\frac{C_1}{a}\right)\cos\theta$$

from which

$$A_1=-\frac{C_1}{a^2}$$

On $r=b$,

$$1000\cos\theta=\left(A_1b+\frac{C_1}{b}\right)\cos\theta$$

from which

$$A_1b+\frac{C_1}{b}=1000$$

or

$$-\frac{C_1}{a^2}b+\frac{C_1}{b}=1000$$

or

$$C_1\frac{a^2-b^2}{a^2b}=1000$$

from which

$$C_1=-\frac{1000\,a^2b}{b^2-a^2}$$

Use of equations (E4) and of the preceding value of C_1 in equations (E3) shows that the components of stress due only to the temperature distribution specified by $T=0$ on $r=a$ and $T=1000\cos\theta$ on $r=b$ are

$$\left.\begin{aligned}\sigma_r&=\frac{1000\,\alpha E}{2(1-\nu)}\frac{a^2b}{r^3}\frac{(b^2-r^2)(r^2-a^2)}{b^4-a^4}\cos\theta\\ \sigma_\theta&=\frac{1000\,\alpha E}{2(1-\nu)}\frac{a^2b}{r^3}\frac{4a^2r^2-(3r^2-b^2)(r^2+a^2)}{b^4-a^4}\cos\theta\end{aligned}\right\}\tag{E5}$$

Radial and tangential stresses were superposed from stresses calculated according to equations (E1) and (E5) to give the exact stresses listed in table II.

APPENDIX F

DETAILS OF RELAXATION CALCULATION OF TANGENTIAL STRESS IN CONCENTRIC CYLINDER WITH SYMMETRICAL TEMPERATURE DISTRIBUTION

The temperature distribution was assumed to be determined by temperatures of 0° F on the internal boundary of 4-inch radius and 500° F on the external boundary of 12-inch radius. The relaxation solution for the temperature distribution is given in figure 6. The values of the elastic constants were assumed to be the same as those used in appendix C, namely,

$$\alpha=8.0\times10^{-6}\ \text{(in./in./°F)}$$

$$E=17.5\times10^{6}\ \text{(lb/sq in.)}$$

$$\nu=0.3$$

As in the asymmetrical case treated in appendix C, the indices of equations (21) to (24) were taken as

$$i=k=n=1$$

$$j=1,2,3$$

and these equations were modified as follows:

On the exterior boundary,

$$\phi_{11}=\phi_{12}=\phi_{13}=\frac{\partial\phi_{11}}{\partial n}=\frac{\partial\phi_{12}}{\partial n}=\frac{\partial\phi_{13}}{\partial n}=0$$

On the interior boundary,

$$\phi_{11}=250x \qquad \frac{\partial\phi_{11}}{\partial n}=250\frac{dy}{ds}$$

$$\phi_{12}=250y \qquad \frac{\partial\phi_{12}}{\partial n}=-250\frac{dx}{ds}$$

$$\phi_{13}=1000 \qquad \frac{\partial\phi_{13}}{\partial n}=0$$

These boundary values are the same as those used for the preceding problem involving the asymmetrical temperature distribution. The biharmonic functions ϕ_{11}, ϕ_{12}, and ϕ_{13} are presented in figures 2 to 4. Calculation of the contour integrals by the methods described in appendix C and solution of equations (26) to (28) show that

$$a_{11}=a_{12}=0$$

and that the values of a_{13} for paths a, b, and c are as given in table III. The values of Airy's stress function as determined entirely by the numerical method were then calculated using the relaxation values of ϕ_{13} and the average of a_{13} for paths a, b, and c according to

$$\phi=a_{13}\,\phi_{13}$$

Values so calculated are presented in table IV. Tangential stresses were calculated from the relaxation solution for the stress function by taking second derivatives with respect to radius according to the formulas of appendix D. These results are also presented in table IV.

APPENDIX G

EXACT DETERMINATION OF AIRY'S STRESS FUNCTION FOR CONCENTRIC CYLINDER WITH SYMMETRICAL TEMPERATURE DISTRIBUTION

The purpose of this section is to present the formula used in calculating Airy's stress function for the concentric circular cylinder with a symmetrical temperature distribution. A possible form for the stress function is given in reference 8 on page 55 as

$$\phi=A\ln r+Br^2\ln r+Cr^2+D \tag{G1}$$

where A, B, C, and D are constants and the corresponding stress components are given by

$$\left.\begin{aligned}\sigma_r&=\frac{A}{r^2}+B(1+2\ln r)+2C\\ \sigma_\theta&=-\frac{A}{r^2}+B(3+2\ln r)+2C\end{aligned}\right\} \tag{G2}$$

The thermal stresses are given by equations (E1). For comparison with the first of equations (G2), the first of equations (E1) is written

$$\sigma_r=\left[\frac{\alpha ET_i}{2(1-\nu)\ln\frac{b}{a}}\frac{a^2b^2}{b^2-a^2}\ln\frac{b}{a}\right]\frac{1}{r^2}+\frac{\alpha ET_i}{2(1-\nu)\ln\frac{b}{a}}\frac{1}{2}(1+2\ln r)+\frac{\alpha ET_i}{2(1-\nu)\ln\frac{b}{a}}\left[-\frac{1}{2}-\ln b-\frac{a^2}{b^2-a^2}\ln\frac{b}{a}\right] \tag{G3}$$

Comparison of equation (G3) with the first of equations (G2) shows that necessary conditions on the constants of equation (G1) are

$$A=\frac{\alpha ET_i}{2(1-\nu)\ln\frac{b}{a}}\frac{a^2b^2}{b^2-a^2}\ln\frac{b}{a}$$

$$B=\frac{\alpha ET_i}{2(1-\nu)\ln\frac{b}{a}}\frac{1}{2}$$

$$C=\frac{\alpha ET_i}{2(1-\nu)\ln\frac{b}{a}}\frac{1}{2}\left(-\frac{1}{2}-\ln b-\frac{a^2}{b^2-a^2}\ln\frac{b}{a}\right)$$

Because the stresses are calculated from derivatives of Airy's stress function, the value of the constant D in equation (G1) may be taken arbitrarily. The values of A, B, and C as just determined are substituted in equation (G1) to obtain

$$\phi=\frac{\alpha E T_i}{2(1-\nu)\ln\frac{b}{a}}\left[-\frac{1}{2}r^2\ln\frac{b}{r}-\frac{1}{4}r^2-\frac{a^2}{b^2-a^2}\left(\frac{r^2}{2}-b^2\ln r\right)\ln\frac{b}{a}\right]+D \tag{G4}$$

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TABLE I—CONTOUR INTEGRALS ALONG PATH c

(a) Integrals involving $\nabla^2\phi_{11}$

j	$\nabla^2\phi_{j+\frac{1}{2}}$	$\nabla^2\phi_{j-\frac{1}{2}}$	$\frac{d\nabla^2\phi_j}{ds}$	$\nabla^2\phi_{n+\frac{1}{2}}$	$\nabla^2\phi_{n-\frac{1}{2}}$	$\frac{\partial\nabla^2\phi_j}{\partial n}$	x	$x\frac{d\nabla^2\phi_j}{ds}$	$x\frac{\partial\nabla^2\phi_j}{\partial n}$	y	$y\frac{d\nabla^2\phi_j}{ds}$	$y\frac{\partial\nabla^2\phi_j}{\partial n}$
1	−113	−113	0	−96	−170	74	5.5	0	407.0	0	0	0
2	−103	−113	10	−81	−105	24	5.5	55.0	132.0	1	10.0	24.0
3	−106	−103	−3	−83	−142	59	5.5	−16.5	324.5	2	−6.0	118.0
4	−85	−106	21	−71	−127	56	5.5	115.5	308.0	3	63.0	108.0
5	−57	−85	28	−51	−92	41	5.5	154.0	225.5	4	112.0	164.0
6	−29	−57	28	−26	−59	33	5.5	154.0	181.5	5	140.0	165.0
7	−52	−29	−23	−27	−59	32	5	−115.0	160.0	5.5	−126.5	176.0
8	−66	−52	−14	−42	−82	40	4	−56.0	160.0	5.5	−77.0	220.0
9	−64	−66	2	−49	−92	43	3	6.0	129.0	5.5	11.0	236.5
10	−38	−64	26	−43	−73	30	2	52.0	60.0	5.5	143.0	165.0
11	−9	−38	29	−18	−19	1	1	29.0	1.0	5.5	159.5	5.5
12	9	−9	18	0	0	0	0	0	0	5.5	99.0	0
13	38	9	29	18	19	−1	−1	−29.0	1.0	5.5	159.5	−5.5
14	64	38	26	43	73	−30	−2	−52.0	60.0	5.5	143.0	−165.0
15	66	64	2	49	92	−43	−3	−6.0	129.0	5.5	11.0	−236.5
16	52	66	−14	42	82	−40	−4	56.0	160.0	5.5	−77.0	−220.0
17	29	52	−23	27	59	−82	−5	115.0	160.0	5.5	−120.5	−176.0
18	57	29	28	26	59	−33	−5.5	−154.0	181.5	5	140.0	−165.0
19	85	57	28	51	92	−41	−5.5	−154.0	225.5	4	112.0	−164.0
20	106	85	21	71	127	−56	−5.5	−115.5	308.0	3	63.0	−108.0
21	103	106	−3	83	142	−59	−5.5	16.5	324.5	2	−6.0	−118.0
22	113	103	10	81	105	−24	−5.5	−55.0	132.0	1	10.0	−24.0
23	113	113	0	96	170	−74	−5.5	0	407.0	0	0	0
24	103	113	−10	81	105	−24	−5.5	55.0	132.0	−1	10.0	24.0
25	106	103	3	83	142	−59	−5.5	−16.5	324.5	−2	−6.0	118.0
26	85	106	−21	71	127	−56	−5.5	115.5	308.0	−3	63.0	108.0
27	57	85	−28	51	92	−41	−5.5	154.0	225.5	−4	112.0	164.0
28	29	57	−28	26	59	−33	−5.5	154.0	181.5	−5	140.0	165.0
29	52	29	23	27	59	−32	−5	−115.0	160.0	−5.5	−126.5	176.0
30	66	52	14	42	82	−40	−4	−56.0	160.0	−5.5	−77.0	220.0
31	64	66	−2	49	92	−43	−3	6.0	129.0	−5.5	11.0	236.5
32	38	64	−26	43	73	−30	−2	52.0	60.0	−5.5	143.0	165.0
33	9	38	−29	18	19	−1	−1	29.0	1.0	−5.5	159.5	5.5
34	−9	9	−18	0	0	0	0	0	0	−5.5	99.0	0
35	−38	−9	−29	−18	−19	1	1	−29.0	1.0	−5.5	159.5	−5.5
36	−64	−38	−26	−43	−73	30	2	−52.0	60.0	−5.5	143.0	−105.0
37	−66	−64	−2	−49	−92	43	3	−6.0	129.0	−5.5	11.0	−236.5
38	−52	−66	14	−42	−82	40	4	56.0	160.0	−5.5	−77.0	−220.0
39	−29	−52	23	−27	−59	32	5	115.0	160.0	−5.5	−120.5	−176.0
40	−57	−29	−28	−26	−59	33	5.5	−154.0	181.5	−5	140.0	−165.0
41	−85	−57	−28	−51	−92	41	5.5	−154.0	225.5	−4	112.0	−164.0
42	−106	−85	−21	−71	−127	56	5.5	−115.5	308.0	−3	63.0	−108.0
43	−103	−106	3	−83	−142	59	5.5	16.5	324.5	−2	−6.0	−118.0
44	−113	−103	−10	−81	−105	24	5.5	−55.0	132.0	−1	10.0	−24.0
Total						0		0	7,540.0		1,914.0	0

TABLE I—CONTOUR INTEGRALS ALONG PATH c—Concluded

(b) Integrals involving T

j	$T_{j+\frac{1}{2}}$	$T_{j-\frac{1}{2}}$	$\frac{dT_j}{ds}$	$T_{n+\frac{1}{2}}$	$T_{n-\frac{1}{2}}$	$\frac{\partial T_j}{\partial n}$	x	$x\frac{dT}{ds}$	$x\frac{\partial T}{\partial n}$	y	$y\frac{dT}{ds}$	$y\frac{\partial T}{\partial n}$
1	387	387	0	491	266	225	3.5	0	1,237.5	0	0	0
2	418	387	31	504	286	218	5.5	170.5	1,199.0	1	31.0	218.0
3	472	418	54	540	342	198	5.5	297.0	1,089.0	2	108.0	396.0
4	539	472	67	592	416	176	5.5	368.5	968.0	3	201.0	528.0
5	609	539	70	652	496	156	5.5	385.0	858.0	4	280.0	624.0
6	676	609	67	714	573	141	5.5	368.5	775.5	5	335.0	705.0
7	544	676	−132	645	573	72	5	−660.0	360.0	5.5	−726.0	396.0
8	422	544	−122	522	438	84	4	−488.0	336.0	5.5	−671.0	462.0
9	315	422	−107	411	316	95	3	−321.0	285.0	5.5	−588.5	522.5
10	223	315	−87	316	216	100	2	−174.0	200.0	5.5	−478.5	550.0
11	166	228	−62	239	143	96	1	−62.0	96.0	5.5	−341.0	528.0
12	123	166	−43	181	99	82	0	0	0	5.5	−236.5	451.0
13	90	123	−33	137	74	63	−1	33.0	−63.0	5.5	−181.5	346.5
14	59	90	−31	93	53	45	−2	62.0	−90.0	5.5	−170.5	247.5
15	22	59	−37	58	26	32	−3	111.0	−96.0	5.5	−203.5	176.0
16	−22	22	−44	13	−10	23	−4	176.0	−92.0	5.5	−242.0	126.5
17	−71	−22	−49	−36	−54	18	−5	245.0	−90.0	5.5	−269.5	99.0
18	−85	−71	−14	−105	−54	−51	−5.5	77.0	280.5	5	−70.0	−255.0
19	−94	−85	−9	−115	−67	−48	−5.5	49.5	264.0	4	−36.0	−192.0
20	−97	−94	−3	−121	−72	−49	−5.5	16.5	269.5	3	−9.0	−147.0
21	−96	−97	1	−123	−71	−52	−5.5	−5.5	286.0	2	2.0	−104.0
22	−93	−96	3	−122	−66	−56	−5.5	−16.5	308.0	1	3.0	−56.0
23	−93	−93	0	−121	−63	−58	−5.5	0	319.0	0	0	0
24	−96	−93	−3	−122	−66	−56	−5.5	16.5	308.0	−1	3.0	56.0
25	−97	−96	−1	−123	−71	−52	−5.5	5.5	286.0	−2	2.0	104.0
26	−94	−97	3	−121	−72	−49	−5.5	−16.5	269.5	−3	−9.0	147.0
27	−85	−94	9	−115	−67	−48	−5.5	−49.5	264.0	−4	−36.0	192.0
28	−71	−85	14	−105	−54	−51	−5.5	−77.0	280.5	−5	−70.0	255.0
29	−22	−71	49	−36	−54	18	−5	−245.0	−90.0	−5.5	−269.5	−99.0
30	22	−22	44	13	−10	23	−4	−176.0	−92.0	−5.5	−242.0	−126.5
31	59	22	37	58	26	32	−3	−111.0	−96.0	−5.5	−203.5	−176.0
32	90	59	31	93	53	45	−2	−62.0	−90.0	−5.5	−170.5	−247.5
33	123	90	33	137	74	63	−1	−33.0	−63.0	−5.5	−181.5	−346.5
34	166	123	43	181	99	82	0	0	0	−5.5	−236.5	−451.0
35	228	166	62	239	143	96	1	62.0	96.0	−5.5	−341.0	−528.0
36	315	228	87	316	216	100	2	174.0	200.0	−5.5	−478.5	−550.0
37	422	315	107	411	316	95	3	321.0	285.0	−5.5	−588.5	−522.5
38	544	422	122	522	438	84	4	488.0	336.0	−5.5	−671.0	−462.0
39	676	544	132	645	573	72	5	660.0	360.0	−5.5	−726.0	−396.0
40	609	676	−67	714	573	141	5.5	−368.5	775.5	−5	335.0	−705.0
41	539	609	−70	652	496	156	5.5	−385.0	858.0	−4	280.0	−624.0
42	472	539	−67	592	416	176	5.5	−368.5	968.0	−3	201.0	−528.0
43	418	472	−54	540	342	198	5.5	−297.0	1,089.0	−2	108.0	−396.0
44	387	418	−31	504	286	218	5.5	−170.5	1,199.0	−1	31.0	218.0
Total						2,853		0	15,843.5		−6,527.0	0

TABLE II—STRESS ALONG POSITIVE X-AXIS IN CONCENTRIC CIRCULAR CYLINDER WITH ASYMMETRICAL TEMPERATURE DISTRIBUTION

Radius (in.)	Stress function	Relaxation tangential stress (lb/sq in.)	Exact stresses (lb/sq in.)		Error[a] in relaxation stress (lb/sq in.)
			Radial	Tangential	
4	−684,900	121,012	0	126,988	−5,976
5	−652,832	65,289	18,126	69,800	−4,511
6	−554,142	25,558	22,172	35,409	−9,851
7	−428,561	2,370	20,962	11,675	−9,305
8	−299,828	−11,372	17,673	−6,277	5,095
9	−182,297	−23,815	13,496	−20,741	3,074
10	−88,278	−32,635	9,013	−32,929	−294
11	−26,440	−35,852	4,473	−43,542	−7,690
12	0	−33,617	0	−53,012	−19,395

[a] Positive signs denote relaxation values larger in absolute magnitude than exact values; negative signs denote relaxation values smaller in absolute magnitude than exact values.

TABLE III—VALUES OF a_{13} AND ASSOCIATED ERRORS

	a_{13}	Error (percent)
Exact	−467.627	------
Path a	−555.0	18.7
Path b	−423.8	9.4
Path c	−416.6	10.9
Average of a, b, c	−465.1	.5

TABLE IV—NUMERICALLY CALCULATED STRESSES AND ASSOCIATED ERRORS FOR CONCENTRIC CYLINDER WITH SYMMETRICAL TEMPERATURE DISTRIBUTION

Radius (in.)	Exact values of Airy's stress function	Relaxation values of Airy's stress function	Exact values of tangential stress (lb/sq in.)	Five-point method for tangential stress using relaxation values of Airy's stress function (lb/sq in.)		Five-point method for tangential stress using exact values of Airy's stress function (lb/sq in.)		Five-point method for tangential stress using Airy's function exact to three figures (lb/sq in.)	
				Stress	Error[a]	Stress	Error[a]	Stress	Error[a]
4	−467,027	−465,100	66,968	81,357	14,369	62,620	−4,368	60,600	−6,400
5	−439,879	−438,124	36,582	42,286	5,704	36,989	407	36,600	0
6	−374,651	−367,894	17,284	14,844	−2,440	17,254	−30	17,600	300
7	−291,678	−281,851	3,636	−620	−4,256	3,627	−9	3,800	200
8	−204,799	−195,807	−6,746	−8,449	1,703	−6,766	20	−7,300	600
9	−124,507	−118,135	−15,058	−15,969	911	−15,054	−4	−13,800	−1,300
10	−59,152	−56,277	−21,964	−21,588	−376	−21,965	1	−23,400	1,400
11	−15,677	−15,813	−27,864	−24,845	−3,019	−27,883	19	−27,400	−500
12	0	0	−33,012	−25,780	−7,232	−32,776	−236	−26,400	−6,600

[a] Positive quantities denote approximate values larger in absolute magnitude than exact values; negative quantities denote approximate values smaller in absolute magnitude than exact values.